Math 261B Tues Sept. 1

$$\overbrace{}^{\text{reductive}} \quad \text{unipotent radical}$$

$$G \supset G_0 \supset G_{lin} \supset R \supset R_u \supset \{e\}$$

finite; e component; Abelian Variety; semisimple; "torus" $(K^\times)^n$

Look at $G = G_{lin}$ in more detail

linear $\Leftrightarrow$ affine

$G \hookrightarrow Gl_n \Rightarrow G \hookrightarrow K^n$ (with polynomial equations)

$A = \mathcal{O}(G)$ ring of polynomial functions

$G \subset K^n \leftarrow x_1, \dots, x_n$

$K[x_1, \dots, x_n] / I(G)$

$= \mathcal{O}(G)$

$I(G)$: pols that vanish on G.

$$\begin{array}{ccc} K^m & & K^n \\ \cup & & \cup \\ X & \longrightarrow & Y \\ \mathcal{O}(X) & \xleftarrow{f} & \mathcal{O}(Y) \end{array}$$

$f(x) \leftarrow y_i$ coords pols $p_i(x_1, \dots, x_n)$

$K[x_1,\dots,x_n]/I(X)$ $\quad$ $K[y_1,\dots,y_n]/I(Y)$

$p_i(x_1,\dots,x_n) \longleftarrow y_i$

X, Y $\quad$ $\mathcal{O}(X\times Y) = \underset{A}{\mathcal{O}(X)} \otimes_k \underset{B}{\mathcal{O}(Y)}$?

$G\times G \xrightarrow{M} G$

$\mathcal{O}(G\times G) \longleftarrow A = \mathcal{O}(G)$

$= A\otimes_k A$

$A \otimes_k B$ has basis $a\otimes b$

$a \in$ basis of A

$b \in$ basis of B

$k[x_1,\dots,x_m]/I \otimes k[y_1,\dots,y_n]/J$

$\cong$

$k[x_1,\dots,x_m,y_1,\dots,y_n]/I(X)+J(Y)$

not obviously a $\sqrt{\ }$ ideal

$X\times Y \subset K^m \times K^n$

$A \xrightarrow{\Delta} A\otimes_k A$

coproduct

$G_a = (K,+)$ $\quad$ $A = k[z]$ $\quad$ $A\otimes A = k[x,y] = k[z]\otimes k[z]$

$k[z] \xrightarrow{\Delta} k[x,y]$ $\quad$ $(x,y) \longmapsto x+y$

$z \longmapsto x+y$ $\quad$ $x = z\otimes 1$ $\quad$ $z^k \otimes z^l$

$y = 1\otimes z$ $\quad$ $x^k y^l$

$z \mapsto z\otimes 1 + 1\otimes z$ $\quad$ (z is "primitive")

$\mathbb{G}_m = (k^\times, \cdot)$ $\quad A = k[t^{\pm 1}]$ $\quad A \otimes A = k[u^{\pm 1}, v^{\pm 1}]$

$t \mapsto uv$

$t \mapsto t \otimes t$ $\quad u = t \otimes 1$, $v = 1 \otimes t$

(t is "grouplike")

$A \xrightarrow{\Delta} A \otimes A$

$z \mapsto \Delta(z) = 1 \otimes z + z \otimes 1$

z is primitive

$\Delta(z^2) = 1 \otimes z^2 + 2z \otimes z + z^2 \otimes 1$

$Gl_n(K)$

$A = K[z_{1,1}, \dots, z_{n,n}, \det(z)^{-1}]$

$\quad = k(z_{ij}, v) / (v \det(z) - 1)$

kG for G finite

dual to $\mathcal{O}(G)$

is a Hopf algebra

$g \in G$ are grouplike

$\Delta g = g \otimes g$

$A \otimes A = K[x_{1,1}, \dots, x_{n,n}, \det(x)^{-1},$
$y_{11}, \dots, y_{nn}, \det(y)^{-1}]$

$A \xrightarrow{\Delta} A \otimes A$

$x_{ij} \mapsto (X \cdot Y)_{ij}$ $\quad \sum_k x_{ik} y_{kj}$

$\det(z)^{-1} \mapsto \det(x)^{-1} \det(y)^{-1}$

$\det(z)$ and $\det(z)^{-1}$ are grouplike.

$G \to \mathbb{G}_m$

$$G \times G \xrightarrow{\mu} G \qquad A \xrightarrow{\Delta} A \otimes A \qquad \text{coproduct}$$

$$\{\circ\} \xrightarrow{e} G \qquad A \xrightarrow{\varepsilon} K \qquad \text{counit}$$

$$G \xrightarrow[g \mapsto g^{-1}]{i} G \qquad A \xrightarrow{S} A \qquad \text{antipode}$$

$\mathcal{O}(G) \longleftarrow k[t^{\pm 1}]$

$\Delta t = t \otimes t$

group like $\longleftarrow t$

Axioms

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mu \times id_G"] \arrow[d, "id_G \times \mu"'] \arrow[dr, phantom, "\boxed{\circlearrowleft}"] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"'] & G
\end{tikzcd}$$

$(u \cdot v) w$

$u \cdot (v \cdot w)$

associativity.

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "id_A \otimes \Delta"] \\
A \otimes A \arrow[r, "\Delta \otimes id_A"'] & A \otimes A \otimes A
\end{tikzcd}$$

Δ is $\square$: coassociative

(A, Δ is a coalgebra)

Ex. $\mathbb{G}_a = (k, +)$ $\quad A = k[z]$ $\quad \Delta z = z \otimes 1 + 1 \otimes z$

$$\Delta f = \sum f_{(1)} \otimes f_{(2)}$$

$$\left.\begin{aligned}(id \otimes \Delta) \circ \Delta &= \sum f_{(1)} \otimes f_{(2,1)} \otimes f_{(2,3)} \\ (\Delta \otimes id) \circ \Delta &= \sum f_{(1,1)} \otimes f_{(1,2)} \otimes f_{(2)}\end{aligned}\right\} = \sum f_{(1)} \otimes f_{(2)} \otimes f_{(3)}$$

$$A \otimes A \to A$$
$$f \otimes g \mapsto fg$$

$$\Delta^{(3)} z = z \otimes 1 \otimes 1 + 1 \otimes z \otimes 1 + 1 \otimes 1 \otimes z$$

$$\Delta^{(n)} z = z \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes z \otimes 1 \otimes \cdots \otimes 1 + \cdots$$

$$G \times G \times G \longrightarrow G \qquad (w, x, y) \to w + x + y$$
$$u \quad v \quad w \mapsto u \cdot v \cdot w$$

$$g \mapsto (g, g) \qquad (g^{-1}, g) \qquad g^{-1} \cdot g$$

$$\begin{tikzcd}
G \arrow[r, "\mathrm{diag}"'] \arrow[dr, bend right, "\mathrm{const}"'] & G \times G \arrow[r, "i \times id_G"] & G \times G \arrow[r, "\mu"] & G \\
& \{\circ\} \arrow[urr, bend right, "e"'] & &
\end{tikzcd}$$

$$g^{-1} \cdot g = e$$

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[dr, bend right, "\varepsilon"'] & A \otimes A \arrow[r, "S \otimes idA"] & A \otimes A \arrow[r, "\cdot"] & A \\
& k \arrow[urr, bend right, "1"'] & &
\end{tikzcd}$$

$$f \mapsto \sum f_{(1)} \otimes f_{(2)} \longmapsto \sum S f_{(1)} \otimes f_{(2)}$$
$$\mapsto \sum (S f_{(1)}) f_{(2)} = \varepsilon(f)$$

$$f = t$$

$$\mathbb{G}_m = (k^\times, \cdot) \qquad A = k[t^{\pm 1}] \qquad \Delta t = t \otimes t$$

$$k^\times \qquad S(t) = t^{-1} \qquad t \otimes t$$

$$t \mapsto t^{-1} \qquad \varepsilon(t) = 1 \qquad t^{-1} \cdot t = 1$$

$\mathbb{G}_a = (K, +)$ $\quad A = k[z]$ $\quad \Delta z = z \otimes 1 + 1 \otimes z$

$S(z) = -z$ $\qquad -z \cdot 1 + 1 \cdot z = 0$

$\varepsilon(z) = 0$

Algebra A, coprod Δ, counit ε, antipode S + axioms

$A = \mathcal{O}(X)$ (Defin) Hopf algebra

(finite gen. reduced, commutative K algebra)

$A = \mathcal{O}(G)$ for an alg. group $\quad g \mapsto M$

(algebraic), linear action $\quad (g, v) \mapsto gv = M \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

V basis $v_1, \dots, v_n$

$v = x_1 v_1 + \dots + x_n v_n \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$G \curvearrowright V \cong K^n$ $\quad \rho : G \times V \to V$

$R = \mathcal{O}(V) = K[x_1, \dots, x_n]$

$R \to R \otimes A$ $\qquad = K[V^*]$

$k[V^*] \to k[V^*] \otimes A$ $\qquad G \curvearrowright V \qquad V^* \curvearrowleft G \qquad G \curvearrowright V^*$, $g \cdot \lambda = \lambda \circ g^{-1}$

$x_i \mapsto \left(M \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right)_i = \sum_j M_{ij} x_j$ $\qquad \lambda \cdot g = (\lambda \circ g)(v)$

$= \lambda(gv)$

$M_{ij}(g)$ $\qquad M_{ij} \in A$ $\qquad \lambda \cdot gh = \lambda \cdot g \cdot h$

(g, h, v)

$$V^* \longrightarrow V^* \otimes A$$

(gh, v)

$$\begin{tikzcd} G \times G \times V \arrow[r, "\mu \times 1_V"] \arrow[d, "1_G \times \rho"'] & G \times V \arrow[d, "\rho"] \\ G \times V \arrow[r, "\rho"] & V \end{tikzcd}$$

(g, hv) $\qquad$ $g(hv)$ $\qquad$ $(gh)v$

commute

$\sum x_j \otimes M_{ij}$

$e \in G$ acts as id_V

$$\begin{tikzcd} V^* & V^* \otimes A \\ R \arrow[r] \arrow[d] & R \otimes A \arrow[d] \\ R \otimes A \arrow[r] & R \otimes A \otimes A \end{tikzcd}$$

$V^* \otimes A$

$$\begin{tikzcd} V^* \arrow[r, "\rho"] \arrow[d, "\rho"'] & V^* \otimes A \arrow[d, "\rho \otimes id_A"] \\ V^* \otimes A \arrow[r, "id_{V^*} \otimes \Delta_A"'] & V^* \otimes A \otimes A \end{tikzcd}$$

ρ is a coaction of A on V^*.

V^* is a (right) comodule for A.

$g \in G$

$\downarrow$

$ev_g \in A^*$

$\lambda \in A^*$

$$V^* \xrightarrow{\rho} V^* \otimes A \xrightarrow{id_{V^*} \otimes \lambda} V^* \otimes k = V^*$$

$$\lambda \in A^* \longmapsto \tilde{\lambda} : V^* \to V^*$$

ρ is a coaction $\iff$ V^* becomes a right A^* module.

$G \hookrightarrow A^*$ gives $V^* \curvearrowleft G$

$g \mapsto ev(g)$ $G \curvearrowright V$

V becomes a left A^* module.

$\Delta^* : (A \otimes A)^* \to A^*$

makes A^* an algebra

$$\begin{array}{ccc} A^* \otimes A^* & \hookrightarrow & (A \otimes A)^* \\ & & \downarrow \Delta \\ & & A^* \end{array}$$

$A^* \xrightarrow[\mu^*]{} (A \otimes A)^* \supset A^* \otimes A^*$

Any A comodule is locally finite .... (next time)